The vector calculus we have learnt so far are in Cartesian Coordinate \((x,y,z)\). Most of the things we've done can also be done in the polar, cylindrical, and spherical coordinate as well. In this section, the polar coordinate is a subset of cylindrical coordinate or spherical coordinate when \(z=0\) or \(\phi=0\), respectively.

**Cylindrical Coordinate System** \((r, \theta, z)\) or \((\rho, \phi, h)\)

**Spherical Coordinate System** \((r, \theta, \phi)\) or \((\rho, \theta, \phi)\)

**WARNING!** The notation for *spherical coordinate* used here is consistent with Engineering Mechanics convention. The coordinate \((r, \theta, \phi)\) refers to radius, angle from \(x\)-axis, and angle from \(z\)-axis.

However, standard physics notation (and British texts) uses \(\phi\) to refer to angle from \(z\)-axis and \(\theta\) for angle from \(x\)-axis and write the coordinate as \((r, \phi, \theta)\) (radius, angle from \(z\), angle from \(x\)).

MAPLE, unless otherwise specified, uses \(\theta\) to refer to angle from \(z\)-axis and \(\phi\) for angle from \(x\)-axis (same as what we are using) but write the coordinate as \((r, \phi, \theta)\).
1. SPACE CURVE

1.1 Representation of Curve

A Curve can be represented by many ways such as a parametric equation, nonparametric equation as projections on the surfaces, and as nonparametric equation as intersections of two surfaces.

1.1.1 Curve from Nonparametric Equations (Explicit or implicit Equation)

For cylindrical and spherical coordinates, we usually express $r = f(\theta)$. The other component may be a function of $r$ or $\theta$.

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = f(x) \quad z = g(x)$</td>
<td>$r = f(\theta) \quad z = g(\theta), z = h(r)$</td>
<td>$r = f(\theta) \quad \phi = g(\theta), \phi = h(r)$</td>
</tr>
</tbody>
</table>

1.1.2 Curve from Parametric Equations

Parametric equation uses another variable (called parameter) to express the coordinate function. The curve C can be expressed parametrically as:

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$</td>
<td>$\mathbf{R}(t) = r(t)\mathbf{e}<em>r + \theta(t)\mathbf{e}</em>\theta + z(t)\mathbf{e}_z$</td>
<td>$\mathbf{R}(t) = r(t)\mathbf{e}<em>r + \theta(t)\mathbf{e}</em>\theta + \phi(t)\mathbf{e}_\phi$</td>
</tr>
</tbody>
</table>
### Example: Helix

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(t) = 5 \cdot \cos(t) i + 5 \cdot \sin(t) j + tk$</td>
<td>$R(t) = 5 e_r + t e_\theta + t e_z$</td>
<td>$R(t) = r(t) e_r + \theta(t) e_\theta + \phi(t) e_\phi$</td>
</tr>
<tr>
<td>$plot3d(\langle 5 \cdot \cos(t), 5 \cdot \sin(t), t \rangle, x = 0 .. 2, t = 0 .. 16, axes = normal, scaling = constrained)$</td>
<td>$plot3d(\langle 5, t, t \rangle, x = 0 .. 2, t = 0 .. 16, axes = normal, scaling = constrained, coords = cylindrical)$</td>
<td>$plot3d\left(\left\langle \sqrt{25 + t^2}, t, \arctan\left(\frac{5}{t}\right) \right\rangle, x = 0 .. 2, t = 0 .. 16, axes = normal, scaling = constrained, coords = spherical \right)$</td>
</tr>
</tbody>
</table>

at $t = \frac{\pi}{2}$ we have the point $\left(0, 5, \frac{\pi}{2}\right)$

at $t = \frac{\pi}{2}$ we have the point $\left(5, \frac{\pi}{2}, \frac{\pi}{2}\right)$

at $t = \frac{\pi}{2}$ we have the point $\left(\sqrt{25 + \frac{\pi^2}{4}}, \frac{\pi}{2}, \arctan\left(\frac{10}{\pi}\right)\right) = \left(5.2, \frac{\pi}{2}, 1.26\right)$

In this example, we can see that the easiest way to represent the helix is by using cylindrical coordinate system.
2. SURFACE

2.1 Representation of Surface

2.1.1 Surface from Nonparametric Equations (Explicit or implicit Equation)

Surface from nonparametric equation are expressed in terms of coordinate variables (x, y, z for Cartesian; r, θ, z for Cylindrical; and r, θ, φ for Spherical). It can be explicit or implicit.

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit: ( z = f(x, y) )</td>
<td>( z = f(r, \theta) )</td>
<td>Explicit: ( \phi = f(r, \theta) )</td>
</tr>
<tr>
<td>Implicit: ( f(x, y, z) = 0 )</td>
<td>Implicit: ( f(r, \theta, z) = 0 )</td>
<td>Implicit: ( f(r, \theta, \phi) = 0 )</td>
</tr>
</tbody>
</table>
Example: Hemisphere

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = \sqrt{2^2 - x^2 - y^2}$ or $x^2 + y^2 + z^2 - 2^2 = 0$ and $z \geq 0$</td>
<td>$r^2 + z^2 = 4$</td>
<td>$r = 2$ and $0 \leq \phi \leq \frac{\pi}{2}$</td>
</tr>
<tr>
<td>$\text{implicitplot3d}(x^2 + y^2 + z^2 - 2^2 = 0, x = -3..3, y = -3..3, z = 0..3, \text{scaling} = \text{constrained}, \text{axes} = \text{normal})$</td>
<td>$\text{implicitplot3d}(r^2 + z^2 = 4, r = -3..3, \theta = 0..2\cdot\pi, z = 0..3, \text{scaling} = \text{constrained}, \text{axes} = \text{normal}, \text{coords} = \text{cylindrical})$</td>
<td>$\text{implicitplot3d}(r = 2, r = -3..3, \theta = 0..2\cdot\pi, \phi = 0..\frac{\pi}{2}, \text{scaling} = \text{constrained}, \text{axes} = \text{normal}, \text{coords} = \text{spherical})$</td>
</tr>
</tbody>
</table>

Note: The range of $r$ in the plot command is for the axes scale only.

We can see clearly that the hemisphere can be easily expressed in spherical coordinate.
2.1.2 Surface from Parametric Equations

Similar to a curve, a surface in a 3-space can be represented by a parametric equation. For surface, we need 2 variables (called $u$ and $v$) instead of just one ($t$) as in the case of curve.

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{R}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$</td>
<td>$\mathbf{R}(u, v) = r(u, v)\mathbf{e}<em>r + \theta(u, v)\mathbf{e}</em>\theta + z(u, v)\mathbf{e}_z$</td>
<td>$\mathbf{R}(u, v) = r(u, v)\mathbf{e}<em>r + \theta(u, v)\mathbf{e}</em>\theta + \phi(u, v)\mathbf{e}_\phi$</td>
</tr>
</tbody>
</table>
### Example: Hemisphere

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \mathbf{R}(u, v) = 2 \cos(u) \cos(v) \mathbf{i} + 2 \sin(u) \cos(v) \mathbf{j} + 2 \sin(v) \mathbf{k} ] where ( 0 \leq u \leq 2 \pi, 0 \leq v \leq \frac{\pi}{2} )</td>
<td>[ \mathbf{R}(u, v) = \mathbf{u} \mathbf{e}<em>r + \mathbf{v} \mathbf{e}</em>\theta + \sqrt{4 - u^2} \mathbf{e}_\phi ] where ( 0 \leq u \leq 2, 0 \leq v \leq 2 \pi )</td>
<td>[ \mathbf{R}(u, v) = 2 \mathbf{e}<em>r + u \mathbf{e}</em>\theta + v \mathbf{e}_\phi ] where ( 0 \leq u \leq 2 \pi, 0 \leq v \leq \frac{\pi}{2} )</td>
</tr>
</tbody>
</table>

\[
\text{plot3d} \left( \begin{array}{c} 2 \cos(u) \cos(v), 2 \sin(u) \cos(v), 2 \\
\sin(v) \end{array} \right), u = 0 .. 2 \pi, v = 0 .. \frac{\pi}{2}, \text{scaling} = \text{constrained}, \text{axes} = \text{normal}, \text{coords} = \text{cylindrical} \right)
\]

\[
\text{plot3d} \left( \begin{array}{c} u, v, \sqrt{4 - u^2} \end{array} \right), u = 0 .. 2, v = 0 .. 2 \pi, \text{scaling} = \text{constrained}, \text{axes} = \text{normal}, \text{coords} = \text{spherical} \right)
\]

- \( u \) constant: semicircle in the plane perpendicular to the \( xy \)-plane.
- \( v \) constant: circle in the plane parallel to the \( xy \)-plane

- \( u \) constant: circle in the plane parallel to the \( xy \)-plane
- \( v \) constant: half-semicircle in the plane parallel to the \( xy \)-plane

- \( u \) constant: semicircle in the plane perpendicular to the \( xy \)-plane.
- \( v \) constant: circle in the plane parallel to the \( xy \)-plane
3. AREA AND VOLUME (NONPARAMETRIC)
3.1 Surface Area in a Plane (Polar and Cartesian XY Coordinates)

Recall from the basic calculus that the area is calculated from:

\[
\text{Area} = \iint dA
\]

In cartesian coordinate \( dA = dx \, dy \)

In polar coordinate \( dA = r \, dr \, d\theta \)

**Example: Circle**

Let's proof it by finding the area of the circle with radius 2 between 1st and 2nd quadrant.

\[
\text{Area} = \int_{0}^{\pi} \int_{0}^{2} r \, dr \, d\theta = 2 \pi
\]

The integration in polar coordinate is much easier to do by hand in this case.
Example: Area between the curves

Find the area between the cardioid \( r = 1 - \cos(\theta) \) and the circle with radius 2 in the first and second quadrant.

```plaintext
with(plots):
polarplot(\{1 - \cos(\theta), 2\}, \theta = 0..2\cdot\pi, scaling = constrained)
```

\[
\int_{0}^{\pi/2} \int_{1}^{2} r \, dr \, d\theta = \frac{5}{4} \pi
\]

Note that the area of the circle is \( \int_{0}^{\pi} \int_{0}^{2} r \, dr \, d\theta = 4 \pi \). The area of the cardioid is \( \int_{0}^{\pi/2} \int_{0}^{1 - \cos(\theta)} r \, dr \, d\theta = \frac{3}{2} \pi \). Subtract the area of the cardioid from the circle and divide by 2 (due to symmetry) gives the same result.

3.2 Volume (Cylindrical, Spherical and Cartesian XYZ Coordinates)

Recall from the basic calculus that the volume is calculated from:

\[
\text{Volume} = \iiint 1 \, dV
\]

In cartesian coordinate \( dV = dx \, dy \, dz \)

In cylindrical coordinate \( dV = r \, dr \, d\theta \, dz \)

In spherical coordinate \( dV = r^2 \sin(\phi) \, dr \, d\theta \, d\phi \)
Example: Volume between the cylinder and the cone

Find the volume between the cone and the cylinder as shown in the figure.

This problem is best done in the cylindrical coordinate. The equation for the cylinder is \( r = 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2 \). The parametric equation in polar coordinate (used for plotting) is \( (2, \theta, z); \ 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2 \).

The equation for the cone is \( r = z, r = 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2 \). The parametric equation in polar coordinate (used for plotting) is \( (z, \theta, z); \ 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2 \).

\[
\int_0^2 \int_0^{2\pi} \int_z^2 r \, dr \, d\theta \, dz = \frac{16}{3} \pi
\]

From basic geometry we can find this volume from \( V = \pi \cdot 2^2 \cdot 2 - \frac{1}{3} \cdot \pi \cdot 2^2 \cdot 2 = \frac{16}{3} \pi \)
Example: Volume between two spheres

Volume between the spheres radius 4 and 5 from θ = 0 to θ = $\frac{\pi}{3}$

$\int_{0}^{\frac{\pi}{3}} \int_{0}^{5} r^2 \cdot \sin(\phi) \, dr \, d\phi = \frac{122}{9} \pi$

From basic geometry, we can calculate this from $\frac{1}{6} \cdot \left( \frac{4}{3} \cdot \pi \cdot 5^3 - \frac{4}{3} \cdot \pi \cdot 4^3 \right) = \frac{122}{9} \pi$
4. GRADIENT, DIVERGENCE, CURL

Like cartesian coordinate, there is also gradient, divergence, and curl in the cylindrical and spherical coordinate. The procedure is slightly different.

4.1 Scalar Field and Vector Field

The scalar field and vector field are defined similarly in the three coordinates.

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar Field: ( f(x, y, z) = 0 )</td>
<td>Scalar Field: ( f(r, \theta, z) = 0 )</td>
<td>Scalar Field: ( f(r, \theta, \phi) = 0 )</td>
</tr>
<tr>
<td>Vector Field ( F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k )</td>
<td>Vector Field ( F(r, \theta, z) = f(r, \theta, z)e_r + g(r, \theta, z)e_\theta + h(r, \theta, z)e_z )</td>
<td>Vector Field ( F(r, \theta, \phi) = f(r, \theta, \phi)e_r + g(r, \theta, \phi)e_\theta + h(r, \theta, \phi)e_\phi )</td>
</tr>
</tbody>
</table>
4.2 Gradient of Scalar Field

The gradient of a scalar field $f$ is the vector field given by:

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar Field: $f(x, y, z) = 0$</td>
<td>Scalar Field: $f(r, \theta, z) = 0$</td>
<td>Scalar Field: $f(r, \theta, \phi) = 0$</td>
</tr>
<tr>
<td>$\nabla \phi = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$</td>
<td>$\nabla \phi = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z$</td>
<td>$\nabla \phi = \frac{\partial f}{\partial r} e_r + \frac{1}{r \sin(\phi)} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r} \frac{\partial f}{\partial \phi} e_\phi$</td>
</tr>
</tbody>
</table>

Remember that the gradient transforms scalar field to a vector field.

**Example: Gradient**

```plaintext
with(VectorCalculus):
SetCoordinates(spherical[r, \theta, \phi]):
Gradient(r^2 \cdot \theta + 3 \phi)
2 r \theta e_r + 3 \phi e_\phi + \left(\frac{r}{\sin(\phi)}\right) e_\theta

SetCoordinates(cylindrical[r, \theta, z]):
Gradient(r^2 \cdot \theta + 3 \cdot z)
2 r \theta e_r + (r) e_\theta + 3 e_z
```
## 4.3 Divergence

A Divergence of a vector field for different coordinate systems are as follows:

<table>
<thead>
<tr>
<th>Coordinate System</th>
<th>Vector Field</th>
<th>Divergence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cartesian</strong></td>
<td>( F(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k )</td>
<td>( \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} )</td>
</tr>
<tr>
<td><strong>Cylindrical</strong></td>
<td>( F(r, \theta, z) = f(r, \theta, z)e_r + g(r, \theta, z)e_\theta + h(r, \theta, z)e_z )</td>
<td>( \nabla \cdot F = \frac{1}{r} \frac{\partial (r \cdot f)}{\partial r} + \frac{1}{r} \frac{\partial g}{\partial \theta} + \frac{\partial h}{\partial z} )</td>
</tr>
<tr>
<td><strong>Spherical</strong></td>
<td>( F(r, \theta, \phi) = f(r, \theta, \phi)e_r + g(r, \theta, \phi)e_\theta + h(r, \theta, \phi)e_\phi )</td>
<td>( \nabla \cdot F = \frac{1}{r^2} \frac{\partial (r^2 \cdot f)}{\partial r} + \frac{1}{r \cdot \sin(\phi)} \frac{\partial g}{\partial \theta} + \frac{1}{r \cdot \sin(\phi)} \frac{\partial (\sin(\phi) \cdot h)}{\partial \phi} )</td>
</tr>
</tbody>
</table>

Remember that the divergence transforms vector field to a scalar field.

---

### Example: Divergence

```plaintext
with(VectorCalculus):

Given vector field \( F(r, \theta, \phi) = 2 \cdot r \cdot e_r + \theta \cdot e_\theta + (\theta + 2 \cdot \phi) \cdot e_\phi \)

SetCoordinates(spherical[r, \theta])

Divergence(VectorField(\(2 \cdot r, \theta + 2 \cdot \phi, \theta\)))
```
Given vector field \( \mathbf{F}(r, \theta, z) = 2 \cdot r \cdot \mathbf{e}_r + \theta \cdot \mathbf{e}_\theta + (\theta + 2 \cdot r) \cdot \mathbf{e}_z \)

Set Coordinates (cylindrical \( [r, \theta, z] \)):

\[
\text{Divergence}(\text{VectorField}(\langle 2 \cdot r, \theta, \theta + 2 \cdot z \rangle)) = \frac{1 + 6 \cdot r}{r}
\]

\[
(9.1)
\]

4.3 Curl

Curls of a vector field in different coordinate systems are given as follows:

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector Field ( \mathbf{F}(x, y, z) = f(x, y, z)i + g(x, y, z)j + h(x, y, z)k )</td>
<td>Vector Field ( \mathbf{F}(r, \theta, z) = f(r, \theta, z)\mathbf{e}<em>r + g(r, \theta, z)\mathbf{e}</em>\theta + h(r, \theta, z)\mathbf{e}_z )</td>
<td>Vector Field ( \mathbf{F}(r, \theta, \phi) = f(r, \theta, \phi)\mathbf{e}<em>r + g(r, \theta, \phi)\mathbf{e}</em>\theta + h(r, \theta, \phi)\mathbf{e}_\phi )</td>
</tr>
</tbody>
</table>

Curl:

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial h}{\partial y} & \frac{\partial g}{\partial z} & \frac{\partial f}{\partial x}
\end{vmatrix}k
\]

Curl:

\[
\nabla \times \mathbf{F} = \frac{1}{r^2} \begin{vmatrix}
\mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
\frac{\partial h}{\partial r} & \frac{\partial g}{\partial \theta} & \frac{\partial f}{\partial x}
\end{vmatrix}k
\]

Curl:

\[
\nabla \times \mathbf{F} = \frac{-1}{r^2 \sin(\phi)} \begin{vmatrix}
\mathbf{e}_r & r \cdot \sin(\phi) \mathbf{e}_\theta & \mathbf{e}_\phi \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\
\frac{\partial h}{\partial r} & r \cdot \sin(\phi) \cdot g & r \cdot h
\end{vmatrix}k
\]

Remember that the curl of a vector field gives another vector field.
Example: Curl

with(VectorCalculus):

Given vector field \( \mathbf{F}(r, \theta, \phi) = 2 \cdot r \cdot \mathbf{e}_r + \theta \cdot \mathbf{e}_\theta + (\theta + 2 \cdot \phi) \cdot \mathbf{e}_\phi \)

SetCoordinates(spherical[r, \theta, \phi]):

\[
\text{Curl}(\text{VectorField}(\langle 2 \cdot r, \theta + 2 \cdot \phi, \theta \rangle ))
\]

\[
\left( \frac{r \cos(\phi) \theta - r}{r^2 \sin(\phi)} \right) \mathbf{e}_r - \frac{\theta}{r} \mathbf{e}_\phi + \left( \frac{\theta + 2 \phi}{r} \right) \mathbf{e}_\theta
\] (10.1)

Given vector field \( \mathbf{F}(r, \theta, z) = 2 \cdot r \cdot \mathbf{e}_r + \theta \cdot \mathbf{e}_\theta + (\theta + 2 \cdot \phi) \cdot \mathbf{e}_z \)

SetCoordinates(cylindrical[r, \theta, z]):

\[
\text{Curl}(\text{VectorField}(\langle 2 \cdot r, \theta, 0 + 2 \cdot z \rangle ))
\]

\[
\left( \frac{1}{r} \right) \mathbf{e}_r + \left( \frac{\theta}{r} \right) \mathbf{e}_z
\] (10.2)
4.4 Properties of Gradient, Divergence, and Curl

The following are true for any coordinate systems.

1. Curl of gradient is the zero vector: \( \nabla \times (\nabla \phi) = 0 \)
2. Divergence of a Curl is the number zero: \( \nabla \cdot (\nabla \times F) = 0 \)
3. Let \( \phi \) and \( \psi \) be scalar fields and \( F \) and \( G \) are vector fields

\[
\begin{align*}
\nabla (\phi + \psi) &= \nabla \phi + \nabla \psi \\
\nabla \cdot (F + G) &= \nabla \cdot F + \nabla \cdot G \\
\nabla \times (F + G) &= \nabla \times F + \nabla \times G \\
\n\nabla (\phi \psi) &= \phi \nabla \psi + \psi \nabla \phi \\
\n\nabla (F \cdot G) &= F \times (\nabla \times G) + G \times (\nabla \times F) + (F \cdot \nabla) G - (G \cdot \nabla) F \\
\n\nabla \cdot (\phi F) &= \phi (\nabla \cdot F) + F \cdot (\nabla \phi) \\
\n\nabla \cdot (F \times G) &= G \cdot (\nabla \times F) - F \cdot (\nabla \times G) \\
\n\n\nabla \times (\phi F) &= (\nabla \phi) \times F + \phi (\nabla \times F) \\
\n\n\nabla \times (F \times G) &= (\nabla \cdot G) F - (\nabla \cdot F) G + (G \cdot \nabla) F - (F \cdot \nabla) G 
\end{align*}
\]